

THE STABLE GALOIS CORRESPONDENCE FOR REAL CLOSED FIELDS

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ABSTRACT. In previous work [6], the authors constructed and studied a lift of the Galois correspondence to stable homotopy categories. In particular, if L/k is a finite Galois extension of fields with Galois group G , there is a functor $c_{L/k}^* : \mathrm{SH}^G \rightarrow \mathrm{SH}_k$ from the G -equivariant stable homotopy category to the stable motivic homotopy category over k such that $c_{L/k}^*(G/H_+) = \mathrm{Spec}(L^H)_+$. The main theorem of [6] says that when k is a real closed field and $L = k[i]$, the restriction of $c_{L/k}^*$ to the η -complete subcategory is full and faithful. Here we “uncomplete” this theorem so that it applies to $c_{L/k}^*$ itself. Our main tools are Bachmann’s theorem on the $(2, \eta)$ -periodic stable motivic homotopy category and an isomorphism range for the map $\pi_*^{\mathbb{R}} \mathbb{S}_{\mathbb{R}} \rightarrow \pi_*^{C_2} \mathbb{S}_{C_2}$ induced by C_2 -equivariant Betti realization.

1. INTRODUCTION

In [8], Levine showed that the “constant” functor $c^* : \mathrm{SH} \rightarrow \mathrm{SH}_k$ from the classical stable homotopy category to the motivic stable homotopy category over an algebraically closed field of characteristic zero is a full and faithful embedding. Inspired by his result, in [6] we introduced and studied functors $c_{L/k}^* : \mathrm{SH}^G \rightarrow \mathrm{SH}_k$, where L/k is a Galois extension with Galois group G . We showed that if k is real closed and $L = k[i]$, then *after completing* at a prime p and at η , if $p \neq 2$, the functor $c_{L/k}^*$ is full and faithful. The need for the completion arose from our lack of knowledge about certain homotopy groups of the motivic sphere over \mathbb{R} . In the meantime advances have been made. Ananyevskiy-Levine-Panin [1] established a motivic version of Serre’s finiteness theorem, which in particular implies that $c_{L/k}^*$ is full and faithful after η -completion. The purpose of this paper is to use Bachmann’s recent results [2], about a localization of $\pi_*^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})$, to remove the η -completions in the main theorem of [6].

Theorem 1.1. *Let k be a real closed field and $L = k[i]$ be its algebraic closure. Then the functor*

$$c_{L/k}^* : \mathrm{SH}^{C_2} \rightarrow \mathrm{SH}_k$$

is a full and faithful embedding.

One of the primary tools in the proof of Theorem 1.1 is the C_2 -equivariant Betti realization functor $\mathrm{Re}_B^{C_2} : \mathrm{SH}_{\mathbb{R}} \rightarrow \mathrm{SH}^{C_2}$ which extends the functor taking a smooth \mathbb{R} -scheme to its \mathbb{C} -points with complex conjugation action. In [5], Dugger-Isaksen study the effect of this functor on the bigraded homotopy groups of the real motivic sphere spectrum, proving that $\mathrm{Re}_B^{C_2} : \pi_{m+n\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})_2^{\wedge} \rightarrow \pi_{m+n\sigma}^{C_2}(\mathbb{S}_{C_2})_2^{\wedge}$ is an isomorphism when $m \geq 2n - 5$. We

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use odd-primary Adams spectral sequences and Bachmann's results to produce a range of bigradings in which the integral version of this map is an isomorphism. In particular, we prove the following.

Theorem 1.2 (Theorem 3.11). *The map $\pi_{m+n\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}}) \rightarrow \pi_{m+n\sigma}^{C_2}(\mathbb{S}_{C_2})$ is*

- (i) *an injection if $m = 2n - 6$ and $n \geq 0$, and*
- (ii) *an isomorphism if $m \geq 2n - 5$ and $n \geq 0$.*

Outline. In Section 2 we recall Bachmann's theorem and deduce some consequences for the C_2 -equivariant Betti realization functor and Morel's \pm -splitting of the 2-periodic stable motivic homotopy category. In Section 3 we adapt the methods of [5] to odd-primary Adams spectral sequences. Via an arithmetic fracture square and the results of Section 2, we deduce Theorem 3.11. Finally, in Section 4 we recall how to bootstrap Theorem 3.11 into a proof of Theorem 1.1.

Notation. We use the following notation throughout the paper.

- » k is a field and L/k is a finite Galois extension of fields with Galois group G .
- » SH_k is Voevodsky's stable motivic homotopy category [11]; hom sets in SH_k are denoted $[\ , \]_k$.
- » SH^G is the G -equivariant stable homotopy category in the sense of [9]; hom sets in SH^G are denoted $[\ , \]_G$.
- » $c_{L/k}^* : \mathrm{SH}^G \rightarrow \mathrm{SH}_k$ is the functor induced by the classical Galois correspondence $G/H \mapsto \mathrm{Spec}(L^H)$, constructed in [6, Section 4.3].¹
- » \mathbb{S}_k is the sphere spectrum in SH_k and \mathbb{S}_G is the sphere spectrum in SH^G .
- » For integers m, n , $S^{m+n\alpha} = (S^1)^{\wedge m} \wedge (\mathbb{A}^1 \setminus 0)^{\wedge n}$. If $G = C_2$, $S^{m+n\sigma} = (S^1)^{\wedge m} \wedge (S^\sigma)^{\wedge n}$ where S^σ is the one-point compactification of the real sign representation.
- » For $X \in \mathrm{SH}_k$, $\pi_{m+n\alpha}^k X := [S^{m+n\alpha}, X]_k$ is the $(m+n\alpha)$ -th homotopy group of X .
- » For $X \in \mathrm{SH}^{C_2}$, $\pi_{m+n\sigma}^{C_2} X := [S^{m+n\sigma}, X]_{C_2}$ is the $(m+n\sigma)$ -th homotopy group of X .
- » For $X \in \mathrm{SH}_k$, $\pi_*^k X := \bigoplus_{m \in \mathbb{Z}} \pi_{m+0\alpha} X$ and similarly for $\pi_*^{C_2} Y$ when $Y \in \mathrm{SH}^G$.
- » Given an embedding $\phi : k \hookrightarrow \mathbb{R}$, $\mathrm{Re}_{B,\phi}^{C_2} = \mathrm{Re}_B^{C_2} : \mathrm{SH}_k \rightarrow \mathrm{SH}^{C_2}$ is the C_2 -equivariant Betti realization which extends the functor taking a smooth k -scheme X to $X(\mathbb{C})$ with the conjugation action [6, §4.4].
- » Depending on context, η is either the motivic Hopf map arising from $\mathbb{A}^2 \setminus 0 \rightarrow \mathbb{P}^1$ or the C_2 -equivariant Hopf map.
- » $(\)_2^\wedge$ is the 2-completion functor and $(\)_\eta^\wedge$ is the η -completion functor. If $a = 2$ or η we have $X_a^\wedge = \mathrm{holim} X/a^n$.
- » We write $X[a^{-1}]$ or $X[1/a]$ for the homotopy colimit of $X \xrightarrow{a} X \xrightarrow{a} \dots$. If $X \simeq X[a^{-1}]$ we say that X is *a-periodic*.

2. PRELIMINARIES

The main new input we use in this paper is a recent theorem of Bachmann [2, Theorem 31]. His theorem compares a localization of $\mathrm{SH}_{\mathbb{R}}$ with the classical stable homotopy category. The most convenient form of his result for us is the following recasting.

¹In *loc. cit.* we state that the category $G\mathrm{sSet}$ of G -simplicial sets is equivalent to the category $\mathrm{sPre}(\mathrm{Or}_G)$ of simplicial presheaves on the orbit category. This isn't quite true: $G\mathrm{sSet} \subseteq \mathrm{sPre}(\mathrm{Or}_G)$ is only a full subcategory. (Thanks to Tom Bachmann for drawing our attention to this inaccuracy.) This has no effect on any of the subsequent mathematics in *loc. cit.* because what is used is that the associated homotopy categories are equivalent and this is true by Elmendorf's Theorem.

Theorem 2.1 (Bachmann). *Betti realization induces an equivalence of triangulated categories*

$$\mathrm{Re}_B^{C_2} : \mathrm{SH}_{\mathbb{R}}[1/2, \eta^{-1}] \xrightarrow{\sim} \mathrm{SH}^{C_2}[1/2, \eta^{-1}].$$

Proof. Consider the functor $\mathrm{Re}_{\mathbb{R}} : \mathrm{SH}_{\mathbb{R}} \rightarrow \mathrm{SH}$ which extends the functor $\mathrm{Sm}_{\mathbb{R}} \rightarrow \mathrm{Top}$, sending X to $X(\mathbb{R})$. Consider as well the composite $\Phi^{C_2} \circ \mathrm{Re}_B^{C_2}$, where $\Phi^{C_2} : \mathrm{SH}^{C_2} \rightarrow \mathrm{SH}$ is the geometric fixed points functor. Both $\mathrm{Re}_{\mathbb{R}}$ and $\Phi^{C_2} \circ \mathrm{Re}_B^{C_2}$ preserve homotopy colimits and if $X \in \mathrm{Sm}_{\mathbb{R}}$ and $n \in \mathbb{Z}$, they both send $\Sigma_{\mathbb{P}^1}^n \Sigma_{\mathbb{P}^1}^{\infty} X_+$ to the spectrum $\Sigma^n \Sigma^{\infty} X(\mathbb{R})_+$. It follows that $\mathrm{Re}_{\mathbb{R}} = \Phi^{C_2} \circ \mathrm{Re}_B^{C_2}$ and so we have the commutative triangle of functors

$$\begin{array}{ccc} \mathrm{SH}_{\mathbb{R}}[1/2, \eta^{-1}] & \xrightarrow{\mathrm{Re}_B^{C_2}} & \mathrm{SH}^{C_2}[1/2, \eta^{-1}] \\ & \searrow \mathrm{Re}_{\mathbb{R}} & \downarrow \Phi^{C_2} \\ & & \mathrm{SH}[1/2]. \end{array}$$

Write $\rho : S^0 \rightarrow S^{\sigma}$ for the standard inclusion. Since $\eta^2 \rho = -2\eta$, we have the equivalence of categories $\mathrm{SH}^{C_2}[1/2, \eta^{-1}] \simeq \mathrm{SH}^{C_2}[1/2, \rho^{-1}]$. It follows that Φ^{C_2} is an equivalence $\mathrm{SH}^{C_2}[1/2, \eta^{-1}] \simeq \mathrm{SH}[1/2]$. A specialization of Bachmann's theorem [2, Theorem 31] says that $\mathrm{Re}_{\mathbb{R}}$ in the above diagram is an equivalence. We conclude that $\mathrm{Re}_B^{C_2}$ is an equivalence as well. \square

Recall Morel's \pm -operations in motivic homotopy theory (see [3, Section 16.2]). Let ε denote the stable map induced by the twist isomorphism $S^{\alpha} \wedge S^{\alpha} \simeq S^{\alpha} \wedge S^{\alpha}$. In the C_2 -equivariant setting, let ε denote the twist $S^{\sigma} \wedge S^{\sigma} \simeq S^{\sigma} \wedge S^{\sigma}$, and note that $\mathrm{Re}_B^{C_2}(\varepsilon) = \varepsilon$. In either the motivic or equivariant setting, invert 2 and note that $e_+ := (\varepsilon - 1)/2$ and $e_- := (\varepsilon + 1)/2$ are orthogonal idempotents. Let the operation $()^+$ denote inversion of e_+ and let $()^-$ denote inversion of e_- , i.e., $()^{\pm}$ is the cofiber of the operation e_{\pm} . For any 2-periodic motivic or C_2 -equivariant spectrum X there is a natural splitting $X \simeq X^+ \vee X^-$.

Lemma 2.2. *If X is a 2-periodic motivic or C_2 -equivariant spectrum, then $X^+ \simeq X_{\eta}^{\wedge}$ and $X^- \simeq X[\eta^{-1}]$.*

Proof. We prove the motivic version of this statement, which easily adapts to the C_2 -equivariant setting. Let X be a spectrum such that $X \simeq X[1/2]$. We have $\varepsilon = -1 - \rho\eta$ whence $e_- = \frac{-1}{2}\rho\eta$. Thus inverting e_- inverts both ρ and η . Since $(2 + \rho\eta)\eta = 0$, this is the same as inverting 2 and η , whence $X^- \simeq X[\eta^{-1}]$. Now apply η -completion to the splitting $X \simeq X^+ \vee X^-$ to get $X_{\eta}^{\wedge} \simeq (X^+)_{\eta}^{\wedge} \vee (X^-)_{\eta}^{\wedge}$. Since $X^- \simeq X[\eta^{-1}]$, the second summand is trivial. Since $e_+\eta = 0$, X^+ is η -complete, i.e., $(X^+)_{\eta}^{\wedge} \simeq X^+$. We conclude that $X_{\eta}^{\wedge} \simeq X^+$, as desired. \square

As an interesting corollary (which we will not use in the remainder of this paper) we note the following.

Proposition 2.3. *The natural map $\mathrm{Re}_B^{C_2}((\mathbb{S}_{\mathbb{R}})_{\eta}^{\wedge}) \rightarrow (\mathbb{S}_{C_2})_{\eta}^{\wedge}$ is an equivalence.*

Proof. Let $\mathbb{S} = \mathbb{S}_{\mathbb{R}}$ and η -complete the 2-primary fracture square for \mathbb{S} in order to produce the bicartesian square

$$\begin{array}{ccc} \mathbb{S}_{\eta}^{\wedge} & \longrightarrow & \mathbb{S}[1/2]_{\eta}^{\wedge} \\ \downarrow & & \downarrow \\ \mathbb{S}_{2,\eta}^{\wedge} & \longrightarrow & (\mathbb{S}_2^{\wedge}[1/2])_{\eta}^{\wedge}. \end{array}$$

Applying $\mathrm{Re}_B^{C_2}$ results in a homotopy pullback square which maps to the corresponding fracture square for $(\mathbb{S}_{C_2})_{\eta}^{\wedge}$. The maps between the vertices on the right edge of the square are equivalences by Lemma 2.2. Thus it suffices to show that $\mathrm{Re}_B^{C_2}(\mathbb{S}_{2,\eta}^{\wedge}) \rightarrow (\mathbb{S}_{C_2})_{2,\eta}^{\wedge}$ is an equivalence. This may be checked on the level of Mackey functor homotopy groups by comparing the motivic and C_2 -equivariant Adams spectral sequences as in [6, Proposition 2.4], concluding the proof. \square

3. COMPARING STABLE STEMS

In this section we establish a range of bidegrees in which the map on stable stems $\pi_{m+n\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}}) \rightarrow \pi_{m+n\sigma}^{C_2}(\mathbb{S}_{C_2})$ induced by equivariant Betti realization is an isomorphism.

Recall that Dugger-Isaksen establish a range in which 2-complete stems are isomorphic.

Theorem 3.1 ([5, Theorem 4.1]). *The map $\pi_{m+n\alpha}^{\mathbb{R}}((\mathbb{S}_{\mathbb{R}})_2^{\wedge}) \rightarrow \pi_{m+n\alpha}^{C_2}((\mathbb{S}_{C_2})_2^{\wedge})$ is an isomorphism if $m \geq 2n - 5$ and an injection if $m = 2n - 6$.*

Remark 3.2. There are isomorphisms $\pi_{m+n\alpha}((\mathbb{S}_{\mathbb{R}})_2^{\wedge}) \cong \pi_{m+n\alpha}((\mathbb{S}_{\mathbb{R}})_{2,\eta}^{\wedge})$ for all m, n by [7, Theorem 1]. Similarly there are isomorphisms $\pi_{m+n\sigma}^{C_2}((\mathbb{S}_{C_2})_2^{\wedge}) \cong \pi_{m+n\sigma}^{C_2}((\mathbb{S}_{C_2})_{2,\eta}^{\wedge})$ for all m, n by [6, Theorem 2.10]. Dugger-Isaksen's result can thus equivalently be stated as a comparison between $(2, \eta)$ -complete stable stems.

Recall the discussion of motivic and C_2 -equivariant cobar complexes from [5, §3], noting that all of these constructions may be made with $H\mathbb{F}_p$, p odd, in place of $H\mathbb{F}_2$. The significance of the p -primary cobar complex is that it forms the E_1 -page of the p -primary Adams spectral sequence (in the motivic or equivariant context). In order to concisely express the properties of these spectral sequences, let \mathbb{S} be \mathbb{S}_k (k a field) or \mathbb{S}_{C_2} , let \mathbb{M}_p denote the homology of a point with coefficients in \mathbb{F}_p or \mathbb{F}_p , let \mathcal{A}_p denote the p -primary motivic or C_2 -equivariant dual Steenrod algebra, let C_p^* denote the p -primary motivic or C_2 -equivariant cobar complex, let γ be α or σ , depending on context, and let $\mathrm{Ext}_{\mathcal{A}_p}^{*,*+\gamma}(\mathbb{M}_p, \mathbb{M}_p)$ denote the homology of C_p^* (which is also Ext in the category of \mathcal{A}_p -comodules).

Theorem 3.3 ([7, Theorem 1] and [6, Theorem 2.10]). *The (motivic or C_2 -equivariant) p -primary Adams spectral sequence has E_1 -page C_p^* and E_2 -page $\mathrm{Ext}_{\mathcal{A}_p}^{*,*+\gamma}(\mathbb{M}_p, \mathbb{M}_p)$; it is strongly convergent with*

$$E_2^{s,m+n\gamma} \Longrightarrow \pi_{m-s+n\gamma} \mathbb{S}_{p,\eta}^{\wedge}.$$

When $p = 2$, we have $\mathbb{S}_{2,\eta}^{\wedge} \simeq \mathbb{S}_2^{\wedge}$ as long as $\mathrm{cd}_2(k[i]) < \infty$ (in the motivic case).

The Dugger-Isaksen result is obtained by comparing cobar complexes. We first extend this method to odd p to obtain an isomorphism range on (p, η) -complete stems. We begin by recalling some facts about the motivic and equivariant Steenrod algebras at odd primes. Write $\mathcal{A}_{\mathbb{F}_p}$ for the classical mod- p dual Steenrod algebra. Recall that

$$\mathcal{A}_{\mathbb{F}_p} = \mathrm{Sym}_{\mathbb{F}_p}(\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots)$$

is a free graded commutative algebra, where $|\tau_i| = p^i$ and $|\xi_i| = p^i - 1$.

Recall that $\mathbb{M}_p^{\mathbb{R}} := \bigoplus_{m,n \in \mathbb{Z}} \pi_{m+n\alpha}^{\mathbb{R}} H\mathbb{F}_p = \mathbb{F}_p[\theta]$ where θ has degree $2 - 2\alpha$. This follows from the affirmative resolution of the Bloch-Kato conjecture [13, Theorem 6.1] together with [10, Theorem 7.4]. In the equivariant case we have $\mathbb{M}_p^{C_2} := \bigoplus_{m,n \in \mathbb{Z}} \pi_{m+n\sigma}^{C_2} H\mathbb{F}_p = \mathbb{F}_p[\theta, \theta^{-1}]$, see *e.g.* [4, Theorem 2.8]. The dual motivic Steenrod algebra over \mathbb{R} is equal to

$$\mathcal{A}_{\mathbb{R}} \cong \mathbb{M}_p^{\mathbb{R}} \otimes_{\mathbb{F}_p} \mathcal{A}_{\mathbb{F}_p} = \mathbb{M}_p^{\mathbb{R}}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / (\tau_0^2, \tau_1^2, \dots),$$

where the elements of $\mathcal{A}_{\mathbb{F}_p}$ are considered to be bigraded by assigning weights so that $|\tau_i| = p^i + (p^i - 1)\alpha$ and $|\xi_i| = (p^i - 1) + (p^i - 1)\alpha$, see [12, Remark 12.12].

Similarly, the dual C_2 -equivariant Steenrod algebra is equal to

$$\mathcal{A}_{C_2} \cong \mathbb{M}_p^{C_2} \otimes_{\mathbb{F}_p} \mathcal{A}_{\mathbb{F}_p} = \mathbb{M}_p^{C_2}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / (\tau_0^2, \tau_1^2, \dots),$$

where in this case elements of $\mathcal{A}_{\mathbb{F}_p}$ are considered to be bigraded by assigning weights so that $|\tau_i| = p^i + (p^i - 1)\sigma$ and $|\xi_i| = (p^i - 1) + (p^i - 1)\sigma$.

Equivariant Betti realization $\text{Re}_B^{C_2}$ induces maps $\mathbb{M}_p^{\mathbb{R}} \rightarrow \mathbb{M}_p^{C_2}$ and $\mathcal{A}_{\mathbb{R}} \rightarrow \mathcal{A}_{C_2}$ which have the obvious effects on the above named elements, *i.e.*, $\theta \mapsto \theta$, $\tau_i \mapsto \tau_i$ and $\xi_i \mapsto \xi_i$.

Write \mathbb{M}_p for either $\mathbb{M}_p^{\mathbb{R}}$ or $\mathbb{M}_p^{C_2}$. In both cases, the dual Steenrod algebra is free over \mathbb{M}_p , and a basis is given by monomials $\tau_0^{\epsilon_0} \tau_1^{\epsilon_1} \dots \xi_1^{n_1} \xi_2^{n_2} \dots$, where $\epsilon_i \in \{0, 1\}$ and n_i is a nonnegative integer. We write $\tau^{\epsilon} \xi^n$ for such a monomial.

Lemma 3.4. *Suppose that $|\tau^{\epsilon} \xi^n| = k + \ell\alpha$. Then $k \leq \frac{p}{p-1}\ell + 1$.*

Proof. The bidegree of ξ_i satisfies $k < \frac{p}{p-1}\ell$ and the bidegree of τ_i satisfies $k \leq \frac{p}{p-1}\ell$ provided $i \geq 1$. It follows that if $\epsilon_0 = 0$ then the bidegree of $\tau^{\epsilon} \xi^n$ satisfies $k \leq \frac{p}{p-1}\ell$. If $\epsilon_0 = 1$ then write $\tau^{\epsilon} \xi^n = \tau_0 \tau^{\epsilon'} \xi^n$, where $\epsilon'_0 = 0$. This element thus satisfies the inequality $k \leq \frac{p}{p-1}\ell + 1$. \square

Lemma 3.5. *The map $C_{\mathbb{R}}^f \rightarrow C_{C_2}^f$ is*

- (i) *an injection in all degrees, and*
- (ii) *an isomorphism if $k - f \geq \left\lfloor \frac{p}{p-1}\ell - \frac{4p-2}{p-1} \right\rfloor + 1$.*

Proof. We have that $C_{C_2}^f \cong \mathbb{M}_p^{C_2} \otimes_{\mathbb{M}_p^{\mathbb{R}}} C_{\mathbb{R}}^f$, that $C_{\mathbb{R}}^f$ is free over $\mathbb{M}_p^{\mathbb{R}}$, and that the map $\mathbb{M}_p^{\mathbb{R}} \rightarrow \mathbb{M}_p^{C_2}$ is injective. It follows that $C_{\mathbb{R}}^f \rightarrow C_{C_2}^f$ is injective.

Let $Z = [z_1 | z_2 | \dots | z_f]$ be a cobar element of Adams filtration f , where each z_i is of the form $\tau^{\epsilon} \xi^n$, of bidegree $k_i + \ell_i\alpha$. By Lemma 3.4 we have that $k_i \leq \frac{p}{p-1}\ell_i + 1$. Summing over i , we find that if $|Z| = k + \ell\alpha$, then $k \leq \frac{p}{p-1}\ell + f$. The cokernel of $C_{\mathbb{R}}^f \rightarrow C_{C_2}^f$ consists of elements of the form $\theta^{-n}Z$ for $n \geq 1$. The elements θ^{-n} lie above the line of slope² $(p-1)/p$ passing through θ^{-1} and so we find that these satisfy the inequality $k \leq \frac{p}{p-1}\ell - \frac{4p-2}{p-1}$. It follows that the bidegree of $\theta^{-n}Z$ satisfies the inequality $k \leq \frac{p}{p-1}\ell + f - \frac{4p-2}{p-1}$. Thus the cokernel is zero in bidegrees satisfying $k > \frac{p}{p-1}\ell + f - \frac{4p-2}{p-1}$. \square

Recall the form of the motivic and C_2 -equivariant Adams spectral sequences from Theorem 3.3. Equivariant Betti realization induces a map between these spectral sequences which we can now analyze.

²We use the standard convention in which the k -axis is horizontal and the $\ell\alpha$ -axis is vertical.

Proposition 3.6. *The map $\text{Ext}_{\mathcal{A}_{\mathbb{R}}}^{(f,k+\ell\alpha)}(\mathbb{M}_p^{\mathbb{R}}, \mathbb{M}_p^{\mathbb{R}}) \rightarrow \text{Ext}_{\mathcal{A}_{C_2}}^{(f,k+\ell\alpha)}(\mathbb{M}_p^{C_2}, \mathbb{M}_p^{C_2})$ is an injection if $k - f = \left\lfloor \frac{p}{p-1}\ell - \frac{4p-2}{p-1} \right\rfloor$ and an isomorphism if $k - f \geq \left\lfloor \frac{p}{p-1}\ell - \frac{4p-2}{p-1} \right\rfloor + 1$.*

Proof. This follows from [5, Lemma 3.4] and Lemma 3.5. \square

Lemma 3.7. $\text{Ext}_{\mathcal{A}_{\mathbb{R}}}^{(f,k+\ell\alpha)}(\mathbb{M}_p^{\mathbb{R}}, \mathbb{M}_p^{\mathbb{R}})$ is a finite-dimensional \mathbb{F}_p -vector space for all f, k, ℓ .

Proof. Writing $C_{\mathbb{F}_p}$ for the classical cobar complex, we have $C_{\mathbb{R}} \cong \mathbb{M}_p^{\mathbb{R}} \otimes_{\mathbb{F}_p} C_{\mathbb{F}_p}$. The universal coefficient theorem then yields the isomorphism

$$\text{Ext}_{\mathcal{A}_{\mathbb{R}}}^{*,*+\alpha}(\mathbb{M}_p^{\mathbb{R}}, \mathbb{M}_p^{\mathbb{R}}) \cong \text{Ext}_{\mathcal{A}_{\mathbb{F}_p}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{M}_p^{\mathbb{R}}$$

up to a grading shift, and this is a finite-dimensional \mathbb{F}_p -vector space in all degrees. \square

Theorem 3.8. *The map $\pi_{m+n\alpha}^{\mathbb{R}}((\mathbb{S}_{\mathbb{R}})_{(p,\eta)}^{\wedge}) \rightarrow \pi_{m+n\sigma}^{C_2}((\mathbb{S}_{C_2})_{(p,\eta)}^{\wedge})$ is*

- (i) *an injection if $m = \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor$, and*
- (ii) *an isomorphism if $m \geq \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor + 1$.*

Proof. This follows from Proposition 3.6 using the same argument as in [5, Theorem 4.1]. \square

Remark 3.9. In [5], Dugger-Isaksen establish a second isomorphism range on 2-complete stems. Namely, $\pi_{m+n\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})_2^{\wedge} \rightarrow \pi_{m+n\sigma}^{C_2}(\mathbb{S}_{C_2})_2^{\wedge}$ is an isomorphism if $m+n \leq -1$. A version of this range appears to hold on (p, η) -complete stems, with the difference that the map might be only surjective when $m+n = -1$. However, this second isomorphism range doesn't extend to integral stems and we do not pursue it further here.

We now turn our attention to p -complete spheres.

Proposition 3.10. *The map $\pi_{m+n\alpha}^{\mathbb{R}}((\mathbb{S}_{\mathbb{R}})_p^{\wedge}) \rightarrow \pi_{m+n\sigma}^{C_2}((\mathbb{S}_{C_2})_p^{\wedge})$ is*

- (1) *an injection if $m = \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor$, and*
- (2) *an isomorphism if $m \geq \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor + 1$.*

Proof. Write \mathbb{S} for either of $\mathbb{S}_{\mathbb{R}}$ or \mathbb{S}_{C_2} . If $p = 2$, the statement of the proposition is Dugger-Isaksen's Theorem 3.1, so we can assume p is odd. In this case 2 is invertible in \mathbb{S}/p^r and in \mathbb{S}_p^{\wedge} . By Lemma 2.2 we have that $(\mathbb{S}_p^{\wedge})^+ \simeq \mathbb{S}_{(p,\eta)}^{\wedge}$ and $(\mathbb{S}_p^{\wedge})^- \simeq \mathbb{S}_p^{\wedge}[\eta^{-1}]$ and so we have that

$$\mathbb{S}_p^{\wedge} \simeq \mathbb{S}_{(p,\eta)}^{\wedge} \vee \mathbb{S}_p^{\wedge}[\eta^{-1}].$$

Note that we have an isomorphism $(\mathbb{S}_p^{\wedge})^- \cong \lim_r((\mathbb{S}/p^r)^-)$. It follows from Theorem 2.1 that the map $\pi_{*+\alpha}^{\mathbb{R}}((\mathbb{S}_{\mathbb{R}})_p^{\wedge}[\eta^{-1}]) \rightarrow \pi_{*+\sigma}^{C_2}((\mathbb{S}_{C_2})_p^{\wedge}[\eta^{-1}])$ is an isomorphism. The result thus follows from Theorem 3.8 and the direct sum decomposition of \mathbb{S}_p^{\wedge} above. \square

Theorem 3.11. *The map $\pi_{m+n\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}}) \rightarrow \pi_{m+n\sigma}^{C_2}(\mathbb{S}_{C_2})$ is*

- (i) *an injection if $m = 2n - 6$ and $n \geq 0$, and*
- (ii) *an isomorphism if $m \geq 2n - 5$ and $n \geq 0$.*

Proof. Consider the comparison of long exact sequences of homotopy groups induced by cofiber sequences

$$\mathbb{S} \rightarrow \left(\prod_p \mathbb{S}_p^{\wedge} \right) \vee \mathbb{S}_{\mathbb{Q}} \rightarrow \left(\prod_p \mathbb{S}_p^{\wedge} \right)_{\mathbb{Q}}.$$

obtained from the arithmetic fracture squares for $\mathbb{S}_{\mathbb{R}}$ and \mathbb{S}_{C_2} . It suffices to show that the comparison map $\pi_{m+n\alpha}^{\mathbb{R}}(-) \rightarrow \pi_{m+n\sigma}^{C_2}(-)$ at the middle and the righthand terms are injections if $m = 2n - 6$ and $n \geq 0$ and an isomorphism if $m \geq 2n - 5$ and $n \geq 0$.

The inequalities for the isomorphism range of p -complete stable stems from [Proposition 3.10](#) for odd p are dominated by Dugger-Isaksen's inequalities in [Theorem 3.1](#), for the 2-complete stable stems. It follows that $\pi_{m+n\alpha}^{\mathbb{R}}(\prod_p(\mathbb{S}_{\mathbb{R}})^{\wedge}_p) \rightarrow \pi_{m+n\alpha}^{C_2}(\prod_p(\mathbb{S}_{C_2})^{\wedge}_p)$ is an injection if $m = 2n - 6$ and an isomorphism if $m \geq 2n - 5$. Since we have that the map $\pi_{m+n\alpha}^{\mathbb{R}}(\prod_p(\mathbb{S}_{\mathbb{R}})^{\wedge}_p)_{\mathbb{Q}} \rightarrow \pi_{m+n\alpha}^{C_2}(\prod_p(\mathbb{S}_{C_2})^{\wedge}_p)_{\mathbb{Q}}$ is a filtered colimit of these maps, it too is an injection if $m = 2n - 6$ and an isomorphism if $m \geq 2n - 5$.

By [Theorem 2.1](#) and [Lemma 2.2](#), $\pi_{*+*\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})^{-}_{\mathbb{Q}} \rightarrow \pi_{*+*\alpha}^{C_2}(\mathbb{S}_{C_2})^{-}_{\mathbb{Q}}$ is an isomorphism. By [\[3, Theorems 11 and 16.2.13\]](#), $(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} \simeq \mathrm{H}\mathbb{Q}$, where $\mathrm{H}\mathbb{Q}$ is the rationalized motivic cohomology spectrum. Thus, we have that $\pi_{i+j\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} = 0$ whenever $j > 0$. When $j = 0$, we have that $\pi_i^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} = 0$ if $i \neq 0$ and $\pi_0^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} = \mathbb{Q}$. We have

$$\pi_{i+j\alpha}^{C_2}(\mathbb{S}_{C_2})^+_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & j \text{ is even and } i + j = 0 \\ 0 & \text{else.} \end{cases}$$

Note that if $j \geq 0$, this vanishing region satisfies $i \geq 2j - 5$. The map $\pi_0^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} \rightarrow \pi_0^{C_2}(\mathbb{S}_{C_2})^+_{\mathbb{Q}}$ is an isomorphism. We conclude that $\pi_{m+n\alpha}^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} \rightarrow \pi_{m+n\sigma}^{C_2}(\mathbb{S}_{C_2})^+_{\mathbb{Q}}$ is an injection if $m = 2n - 6$ and $n \geq 0$, and an isomorphism if $m \geq 2n - 5$ and $n \geq 0$. \square

4. PROOF OF [THEOREM 1.1](#)

We finish by explaining how the comparison of stable stems in the previous section implies the embedding theorem.

Proposition 4.1. *If*

- (i) $\mathrm{Re}_B^{C_2} : [S^n, \mathbb{S}_{\mathbb{R}}]_{\mathbb{R}} \xrightarrow{\cong} [S^n, \mathbb{S}_{C_2}]_{C_2}$, and
- (ii) $\mathrm{Re}_B^{C_2} : [\mathrm{Spec}(\mathbb{C})_+ \wedge S^n, \mathbb{S}_{\mathbb{R}}]_{\mathbb{R}} \xrightarrow{\cong} [C_2 + \wedge S^n, \mathbb{S}_{C_2}]_{C_2}$

are isomorphisms for all $n \in \mathbb{Z}$, then [Theorem 1.1](#) is true for any real closed field k .

Proof. Let k be a real closed field and $L = k[i]$. To prove [Theorem 1.1](#), it suffices to prove that

- (a) $c_{L/k}^* : [S^n, \mathbb{S}_{C_2}]_{C_2} \xrightarrow{\cong} [S^n, \mathbb{S}_k]_k$, and
- (b) $c_{L/k}^* : [C_2 + \wedge S^n, \mathbb{S}_{C_2}]_{C_2} \xrightarrow{\cong} [\mathrm{Spec}(L)_+ \wedge S^n, \mathbb{S}_k]_k$

are isomorphisms for all $n \in \mathbb{Z}$, by the same argument as in the beginning of the proof of [\[6, Theorem 2.21\]](#).

To prove that the maps in (a), (b) are isomorphisms, we can assume that $k = \mathbb{R}$ and $L = \mathbb{C}$, by the same argument as in [\[6, Proposition 2.20\]](#). We now consider the C_2 -equivariant Betti realization functor $\mathrm{Re}_B^{C_2} : \mathrm{SH}_{\mathbb{R}} \rightarrow \mathrm{SH}^{C_2}$. Since $\mathrm{Re}_B^{C_2} \circ c_{\mathbb{C}/\mathbb{R}}^* = \mathrm{id}$, it follows that (a) and (b) are isomorphisms. \square

Corollary 4.2 ([Theorem 1.1](#)). *Let k be a real closed field and $L = k[i]$ be its algebraic closure. Then the functor*

$$c_{L/k}^* : \mathrm{SH}^{C_2} \rightarrow \mathrm{SH}_k$$

is a full and faithful embedding.

Proof. If $i < 0$ then $\pi_i^{\mathbb{R}}(\mathbb{S}_{\mathbb{R}}) = \pi_i^{C_2}(\mathbb{S}_{C_2}) = 0$ and so the map in 4.1(i) is an isomorphism for $i < 0$. It is an isomorphism for $i \geq 0$ by setting $n = 0$ in Theorem 3.11. The map in 4.1(ii) is identical to the map $[S^n, \mathbb{S}_{\mathbb{C}}]_{\mathbb{C}} \rightarrow [S^n, \mathbb{S}]$ induced by complex Betti realization. This is an isomorphism by Levine's theorem [8]. \square

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